METU - NCC

CALCULUS FOR FUNCTIONS OF SEVERAL VARIABLES MIDTERM 1	
Code : *MAT 120* Acad.Year: *2014-2015* Semester : *FALL* Date : *25.10.2015* Time : *15:40* Duration : *100 min*	Last Name: Name : KEY Student # : Signature : 7 QUESTIONS ON 5 PAGES TOTAL 100 POINTS

1. (10)	2. (10)	3. (10)	4. (15)	5. (20)	6. (25)	7. (10)		

Please draw a $\boxed{\text{box}}$ around your answers. No calculators, cell-phones, notes, etc. allowed.

1. *(10pts)* Show that the vectors $a = \langle 1, 1, 1 \rangle$, $b = \langle 1, 0, 1 \rangle$ and $c = \langle 0, 0, -1 \rangle$ are not coplanar (i.e., do not lie on the same plane).

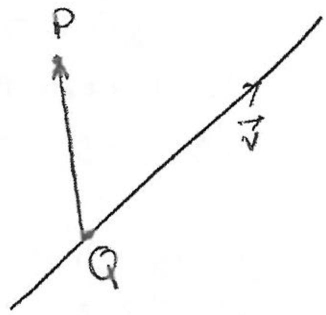
Calculate the box product $\vec{a} \cdot (\vec{b} \times \vec{c}) =$

$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{vmatrix} = 1$, that is, the volume V of the box generated by $\vec{a}, \vec{b}, \vec{c}$ is not zero. Whence these vectors are not coplanar.

2. *(10pts)* Find an equation of the plane passing through the point $P(2, 1, -1)$ and containing the line $x = 1 - t, y = 1 + t, z = 2t$.

Take a point from the line, say $Q(1,1,0)$ $(t=0)$.

Then $\vec{n} = \overrightarrow{QP} \times \vec{v}$ is the normal vector of the plane sought, where $\vec{v} = \langle -1, 1, 2 \rangle$ is the vector of the line. But



$\overrightarrow{QP} = \langle 1, 0, -1 \rangle$ and

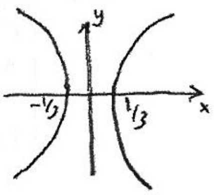
$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ -1 & 1 & 2 \end{vmatrix} = \langle 1, -1, 1 \rangle$. Hence $(x-2) - (y-1) + (z+1) = 0$ or $x - y + z = 0$

is the equation of the plane.

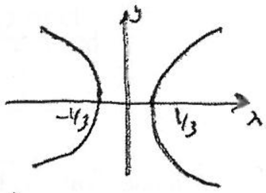
3. *(5+5=10pts)* Consider the surface $3y^2 - 9x^2 + z^2 + 1 = 0$.

(a) Sketch and name the curves found by intersecting the surface with the xy-plane, xz-plane and the yz-plane.

xy-plane ; $z=0 \Rightarrow 9x^2 - 3y^2 = 1$, $y = \pm\sqrt{3x^2 - 1/3}$, $|x| \geq 1/3$



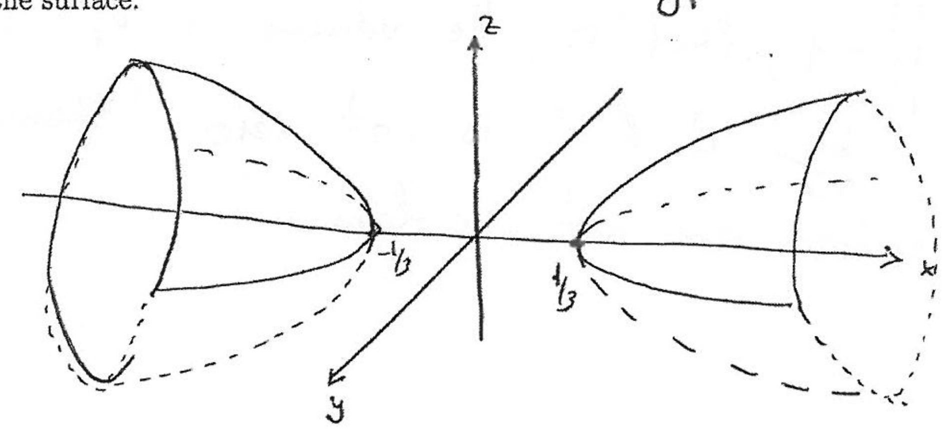
xz-plane ; $y=0 \Rightarrow 9x^2 - z^2 = 1$, $z = \pm\sqrt{9x^2-1}$, $|x| \geq 1/3$



yz-plane ; $x = 0 \Rightarrow 3y^2 + z^2 = -1 \Rightarrow S \cap (yz) = \emptyset$

hyperboloid of two sheets.

(b) Sketch the surface.



4. *(15pts)* Find the distance from the point $P(-6, 1, 21)$ to the line $x = 3t - 4, y = t - 5, z = t - 1$.

I way: Consider the plane through P and $\perp$ to the line: $\vec{v} = \langle 3, 1, 1\rangle$ is the normal of that plane $\Rightarrow 3(x+6) + (y-1) + (z-21) = 0$ or $3x + y + z = 4$ is the plane equation. To find out its intersection with the line, we put $3(3t-4) + (t-5) + (t-1) = 4 \Rightarrow$ $\Rightarrow t = 2 \Rightarrow Q(2, -3, 1) = \Pi \cap \ell$ and $d = d(P, Q) = \sqrt{64 + 16 + 400} =$ $= 4\sqrt{30}$

II way: Take a point from the line, say $Q = Q(-4, -5, -1)$. Then $\vec{QP} = \langle -2, 6, 22\rangle \Rightarrow \vec{QP} \times \vec{v} = 2\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 3 & 11 \\ 3 & 1 & 1 \end{vmatrix} = -4(4\vec{i} - 17\vec{j} + 5\vec{k}) \Rightarrow$

$$\Rightarrow d = \frac{\|\vec{QP} \times \vec{v}\|}{\|\vec{v}\|} = 4\sqrt{4^2 + 17^2 + 5^2}\,\sqrt{11}^{-1} = 4\sqrt{\frac{330}{11}} = 4\sqrt{30}$$

III way: Use projection of $\vec{QP}$ onto $\vec{v}$.

5. *(4×5=20pts)* Find the following limits it they exists or show that they don't exist.

(a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+y^2-1}{2xy\cos(y)+3}$ Since the function $f(x,y)$ is continuous at $(0,0)$, we have $\lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0) = -1/3$

(b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy\cos y}{3x^2+y^2}$ if $x=0$, $f(0,y)=0 \to 0$ as $y\to 0$;

if $y=0$, $f(x,0)=0\to 0$ as $x\to 0$; but along $y=x$ we have $f(x,x)=\dfrac{x^2\cos(x)}{4x^2}=\dfrac{\cos(x)}{4}\to 1/4$ as $x\to 0$. Thereby the limit does not exist.

(c) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+\sin^2 y}{2x^2+y^2}$ if $x=0$, $f(0,y)=\dfrac{\sin^2 y}{y^2}\to 1$ as $y\to 0$;

If $y=0$, $f(x,0)=\dfrac{1}{2}\to 1/2$ as $x\to 0$.

So, again the limit does not exist.

(d) $\lim\limits_{(x,y)\to(0,0)} \dfrac{2y^3\arctan(xy)}{x^2+y^2}$ Take $\varepsilon>0$. Put δ to be ε/π.

Then $\left|\dfrac{2y^3\arctan(xy)}{x^2+y^2}\right| = 2\dfrac{y^2}{x^2+y^2}|y||\arctan(xy)| \le$

$\le 2|y||\arctan(xy)| \le 2|y|\frac{\pi}{2} \le \pi\sqrt{x^2+y^2} \le \pi\delta \le \varepsilon$

whenever $\sqrt{x^2+y^2}\le\delta$. Consequently

$\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0.$

6. *(5×5=25pts)* Find the indicated partial derivatives.

(a) f_x if $f(x,y) = x\ln(xy^2)$, $f_x = \ln(xy^2) + x\,\frac{1}{xy^2}\,y^2 = \ln(xy^2) + 1$

(b) g_y if $g(x,y) = \arctan(2y^2)\sin(x\sqrt{y})$, $g_y = \frac{4y\sin(x\sqrt{y})}{1+4y^4} + \frac{\arctan(2y^2)\cos(x\sqrt{y})\cdot x}{2\sqrt{y}}$

(c) $\frac{\partial z}{\partial t}$ if $z = 2x^2 - \sqrt{y}$ and $x = 2st$, $y = t^2 + 3s$, $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$

$= 4x \cdot 2s + \frac{-1}{2\sqrt{y}} \cdot 2t = 16s^2t - \frac{\cancel{2}t}{\cancel{2}\sqrt{t^2+3s}} =$

$= 16s^2t - \frac{t}{\sqrt{t^2+3s}}$

(d) $h_{xy}(1,0)$ if $h(x,y) = \ln(x^2 - y)$, $h_x = \frac{2x}{x^2-y}$, $h_{xy} = \frac{2x}{(x^2-y)^2}$

So, $h_{xy}(1,0) = \frac{2}{1} = 2$.

(e) $\frac{\partial x}{\partial z}$ if $e^{xy} + \sin(xz) + 1 = 0$. Put $x = x(y,z)$ but locally. Then

$e^{xy} \cdot y \cdot \frac{\partial x}{\partial z} + \cos(xz)\left(\frac{\partial x}{\partial z} \cdot z + x\right) = 0 \Rightarrow$

$\frac{\partial x}{\partial z} = -\frac{x\cos(xz)}{ye^{xy} + z\cos(xz)}$ if $ye^{xy} + z\cos(xz) \neq 0$.

7. *(5+5=10pts)* Consider the vector function $r(t) = \left\langle \frac{\ln(1+t)}{t}, \frac{t^2+1}{t-1}, \left(\frac{1}{2}\right)^t \right\rangle$.

(a) Find the domain of $r(t)$. We have $t > -1$, $t \neq 0$, $t \neq 1$.

Therefore $\operatorname{dom}(r(t)) = (-1,0) \cup (0,1) \cup (1,\infty)$

(a) Find $\lim_{t\to 0} r(t)$.

Since $\lim_{t\to 0} \frac{\ln(1+t)}{t} = \lim_{t\to 0} \frac{1/(1+t)}{1} = 1$ thanks to

L'Hospital Rule from MAT 119,

$\lim_{t\to 0} \frac{t^2+1}{t-1} = -1$, and $\lim_{t\to 0} \left(\frac{1}{2}\right)^t = 1$, it follows

that

$$\lim_{t\to 0} \vec{r}(t) = \langle 1, -1, 1 \rangle \in \mathbb{R}^3.$$